

Optimal Auction Design for Flexible Consumers

Shiva Navabi, *Student Member, IEEE*, Ashutosh Nayyar, *Member, IEEE*,

Abstract—We study the problem of designing revenue-maximizing auctions for allocating multiple goods to flexible consumers. In our model, each consumer is interested in a subset of goods known as its flexibility set and wants to consume one good from this set. A consumer's flexibility set and its utility from consuming a good from its flexibility set are its private information. We focus on the case of nested flexibility sets — each consumer's flexibility set can be one of k nested sets. We provide several examples where such nested flexibility sets may arise. We characterize the allocation rule for an incentive compatible, individually rational and revenue-maximizing auction as the solution to an integer program. The corresponding payment rule is described by an integral equation. We then leverage the nestedness of flexibility sets to simplify the optimal auction and provide a complete characterization of allocations and payments in terms of simple thresholds.

Index Terms—Revenue maximization, Bayesian incentive compatibility, flexible demand, optimal auction.

I. INTRODUCTION

The problem of allocating limited resources among multiple users arises frequently in a wide array of applications ranging from communication networks to transportation and power systems. In many such applications, the users are selfish agents with private information about their preferences and constraints. Finding a desirable allocation of resources would typically require at least a partial knowledge of users' private preferences and constraints. The users, however, can behave strategically in revealing their private information to benefit themselves at the expense of other users and/or the owner of the resources being allocated. Thus, the presence of strategic users with private information creates two key challenges for the resource allocation problem: (i) the allocation needs to be based on the information revealed by the users; (ii) the allocation procedure must anticipate users' strategic behavior in the revelation of their private information. The economic theory of mechanism design provides a framework for addressing such resource allocation problems.

Auctions provide one of the simplest settings of a mechanism design problem. An auctioneer/mechanism designer would typically ask for bids from potential customers, and allocate resources and charge payments as a function of the received bids. Customers with private information about their utilities can be strategic about what bids they submit. The auction design problem is to find suitable allocation and payment functions, which map the customers' bids to allocations and payments, so that the auctioneer can achieve some desired objective. Typically, the auctioneer's objectives are either maximization of its revenue or maximization of social welfare.

In this paper, we consider the problem of designing revenue-maximizing auctions for multiple goods and flexible consumers. Consumer flexibility about goods can arise in different scenarios. In demand response programs of electric utilities, some consumers may be flexible about when and at what rate they receive power. In airline/hotel reservation settings, customers may be flexible about their travel dates. The seller of these goods/services should be able to take this flexibility into account to improve its profits. In our setup, each consumer is associated with a *flexibility set* that describes the subset of goods the consumer is equally interested in. Each consumer wants to consume *one good from its flexibility set*. The flexibility set of a consumer and the utility it gets from consuming a good from its flexibility set are both its private information.

We focus on the case of nested flexibility sets — each consumer's flexibility set can be one of the k sets, $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$, which are nested in the following way:

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots \subset \mathcal{B}_k. \quad (1)$$

If consumers' flexibility sets are truthfully revealed to the auctioneer, the nestedness in (1) allows the auctioneer to compare consumer flexibility and say whether a given consumer is more, less or equally flexible as another consumer.

A. Examples of Nested Flexibility

There are several markets where consumer flexibility resembles the nested pattern in (1). For example, consider flexible electricity consumers that need one unit of energy within a certain deadline [1]. Let \mathcal{B}_τ denote the set of energy units available for delivery in the interval $[0, \tau]$, $\tau = 1, \dots, k$. Clearly, $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots \subset \mathcal{B}_k$. A consumer who needs one unit of energy with a deadline of 2 can be seen as having \mathcal{B}_2 as its flexibility set, that is, it needs one good from \mathcal{B}_2 . A similar flexibility model appears in auctions with deadline-based goods such as airline ticket auctions where different customers may have different departure deadlines.

As another example, consider electricity consumers that need to receive a fixed amount of energy within a fixed time interval while having certain constraints on the rate at which they can receive energy. Suppose each consumer needs to receive one unit of energy within the time interval $[0, T]$ but some consumers need energy at a constant rate while others can tolerate variable rates. Let \mathcal{B}_1 be the set of energy units that the energy provider can supply at a constant rate over the interval $[0, T]$ and \mathcal{B}_2 be the set of all energy units that can be supplied over the interval $[0, T]$. We thus have consumers whose flexibility sets are either \mathcal{B}_1 or \mathcal{B}_2 with $\mathcal{B}_1 \subset \mathcal{B}_2$.

Another example of consumer flexibility comes from auction-based spectrum allocation in cognitive-radio networks ([2], [3], [4]) where a primary spectrum owner has multiple

S. Navabi and A. Nayyar are with the Electrical Engineering Department, University of Southern California, 3740 McClintock Avenue, Los Angeles, CA, 90089 USA (E-mail: navabiso@usc.edu; ashutosn@usc.edu.)

frequency bands with different bandwidths. These bands can be allocated to secondary users who need a certain minimum amount of bandwidth. Suppose the primary owner has frequency bands of widths w_1, w_2, \dots, w_k with $w_1 < w_2 < \dots < w_k$. Let $\mathcal{W}_i, i = 1, 2, \dots, k$, denote the set of frequency bands of width w_i that are available for allocation to secondary users. Define $\mathcal{B}_i = \bigcup_{j=k-i+1}^k \mathcal{W}_j, i = 1, 2, \dots, k$, as the set of frequency bands of width greater than or equal to w_{k-i+1} . We thus have $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots \subset \mathcal{B}_k$. A secondary user that needs one frequency band of width at least w_i can be interpreted as having \mathcal{B}_{k-i+1} as its flexibility set.

Consider next auction-based content delivery in Wireless Information Centric Networks [5] where multiple content providers compete for limited cache storage resources provided by a Wireless Access Point (WAP) in a given region for a certain time period. Suppose the WAP has k cache servers with storage capacities $c_1 < c_2 < \dots < c_k$. Assume that one cache server can serve at most one content provider at a time. Let \mathcal{B}_i be the set of cache servers with capacity greater than or equal to c_{k-i+1} . Clearly the sets $\mathcal{B}_i, i = 1, \dots, k$ are nested. A content provider who needs a cache of storage capacity at least c_{k-i+1} has the flexibility set \mathcal{B}_i .

B. Prior Literature

The problem of designing auctions has been investigated in many different setups in the prior literature. Numerous works have addressed social welfare maximizing or *efficient* auction design, the most well-known of these being the Vickrey-Clarke-Groves (VCG) mechanism ([6], [7], [8]). While VCG auctions have several interesting features including dominant-strategy truthful implementation, there are also several downsides with this framework as pointed in [9], [10] and [11].

Several works have addressed instances of the revenue-maximizing auction design problem with certain assumptions about users' private information and their utility functional forms that make the problem more tractable. In his seminal paper [12], Myerson derived fundamental results for the expected-revenue maximizing single-object auction problem. In sequel, several works extended these classic results to multi-item auction problems. Armstrong [13] extended the analysis to the case where two objects are to be allocated. Hartline et al [14], survey various solution approaches for addressing revenue-optimal auctions in cases where users' private information is unidimensional.

The problem of allocating multiple goods to several customers with special preferences over the set of offered items has long been studied in the context of combinatorial auctions ([15, Chapter 8], [16, Chapter 11], [17], [18]). In auctions that involve sale of heterogeneous goods, customers' preferences for various *subsets* of items may be different from their preferences for those items separately. As a result complementarities and substitution effects between the offered goods arise that can be incorporated to develop more profitable allocations. Several works have studied complementarities by allowing the customers to bid for bundles of items. Avery et al. [19] studied bundling effects in multi-object auctions

under linearity assumptions for utility functions. Some recent works have modeled an extreme case of complementarity among the goods in multi-object auctions by imposing the assumption of having single-minded buyers who are interested in getting all goods from certain subsets of items. Ledyard [20] characterizes a revenue maximizing dominant strategy auction for single-minded buyers. Abhishek and Hajek [21] consider the same problem of revenue optimal auction design for single-minded bidders with the users' preferred bundle being their private information.

C. Organization

The rest of the paper is organized as follows: we discuss the problem formulation and the mechanism setup in Section II. In Section III, we characterize incentive compatibility and individual rationality constraints for the mechanism. We show that the optimal allocation is the solution to an integer program in Section IV. In Section V, we simplify the optimal allocation and payments and characterize them in terms of simple thresholds. We summarize our findings and briefly point out potential extensions to the current framework in Section VI.

D. Notations

$\{0, 1\}^{N \times M}$ denotes the space of $N \times M$ dimensional matrices with entries that are either 0 or 1. \mathbb{Z}^+ is the set of non-negative integers. For a set \mathcal{A} , $|\mathcal{A}|$ denotes the cardinality of \mathcal{A} . x^+ is the positive part of the real number x , that is, $x^+ = \max(x, 0)$. Vector inequalities are component-wise; that is, for two $1 \times n$ dimensional vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{u} \leq \mathbf{v}$ implies that $u_i \leq v_i$, for $i = 1, \dots, n$. $\mathbb{1}_{\{a \leq b\}}$ denotes 1 if the inequality in the subscript is true and 0 otherwise. \mathbb{E} denotes the expectation operator. For a random variable θ , \mathbb{E}_θ denotes that the expectation is with respect to the probability distribution of θ .

II. PROBLEM FORMULATION

We consider a setup where an auctioneer has M goods and N potential customers. $\mathcal{M} = \{1, 2, \dots, M\}$ denotes the set of goods and $\mathcal{N} = \{1, 2, \dots, N\}$ denotes the set of potential customers. Customer $i, i \in \mathcal{N}$, has a flexibility set $\phi_i \subset \mathcal{M}$ which represents the set of goods the customer is equally interested in. Customer i can consume at most one good from its flexibility set ϕ_i . We assume that the flexibility set of each customer can be one of k nested sets. That is, we have k nested subsets of the set of goods:

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots \subset \mathcal{B}_k \subseteq \mathcal{M}, \quad (2)$$

and $\phi_i \in \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k\}$ for every $i \in \mathcal{N}$. If $\phi_i = \mathcal{B}_j$, we say that customer i 's *flexibility level*, denoted by b_i , is j . Customer i 's utility for receiving a good from ϕ_i is θ_i .

We assume that θ_i and b_i are customer i 's private information and are unknown to other users as well as the auctioneer. We assume that $(\theta_i, b_i), i \in \mathcal{N}$, are independent random pairs taking values in the product sets $[\theta_i^{\min}, \theta_i^{\max}] \times$

$\{1, 2, \dots, k\}^1$, $i \in \mathcal{N}$. The probability distributions f_i of (θ_i, b_i) , $i \in \mathcal{N}$, are assumed to be common knowledge. We define $\theta := (\theta_1, \theta_2, \dots, \theta_N)$ and $b := (b_1, b_2, \dots, b_N)$ as the customers' valuations profile and flexibility levels profile, respectively. $f(\theta, b)$ is the joint probability distribution of (θ, b) . Let $\Theta_i := [\theta_i^{\min}, \theta_i^{\max}]$ and $\Theta := \prod_{i=1}^N \Theta_i$. The pair (θ_i, b_i) is referred to as customer i 's type.

An allocation of the goods among the customers can be described by an $N \times M$ dimensional matrix \mathbf{A} with the entries $\mathbf{A}(i, j) = 1$ if customer i gets good j and $\mathbf{A}(i, j) = 0$ otherwise. The matrix \mathbf{A} is called an allocation matrix. We assume that the goods are indexed such that the first $|\mathcal{B}_l|$ goods belong to \mathcal{B}_l , for $l = 1, \dots, k$.

We require that each of the M available goods be allocated to at most one customer and that each customer receives at most one good. This implies that $\sum_{i=1}^N \mathbf{A}(i, j) \leq 1, \forall j$ and $\sum_{j=1}^M \mathbf{A}(i, j) \leq 1, \forall i$. A binary matrix \mathbf{A} that satisfies these two constraints is called a *feasible* allocation matrix. Let $\mathcal{S} \subset \{0, 1\}^{N \times M}$ denote the set of all feasible allocation matrices. That is,

$$\mathcal{S} := \left\{ \mathbf{A} \in \{0, 1\}^{N \times M} \mid \sum_{i=1}^N \mathbf{A}(i, j) \leq 1, \right. \\ \left. \forall j \in \mathcal{M}, \sum_{j=1}^M \mathbf{A}(i, j) \leq 1, \forall i \in \mathcal{N} \right\}. \quad (3)$$

Given an allocation matrix \mathbf{A} and a payment t_i charged to customer i , the net utility for this customer is

$$u_i(\theta_i, b_i, \mathbf{A}, t_i) = \theta_i \left(\sum_{j \in \mathcal{B}_{b_i}} \mathbf{A}(i, j) \right) - t_i. \quad (4)$$

We consider direct mechanisms where, for each $i \in \mathcal{N}$, customer i reports a valuation from the set Θ_i and a flexibility level from the set $\{1, 2, \dots, k\}$ to the auctioneer. The customers can misreport their valuations as well as their flexibility levels. A mechanism consists of an allocation rule q and a payment rule t . The allocation rule q is a function from the type profile space $\Theta \times \{1, 2, \dots, k\}^N$ to the set of feasible allocation matrices \mathcal{S} . The payment rule is a mapping from $\Theta \times \{1, 2, \dots, k\}^N$ to \mathbb{R}^N with the i th component t_i being the payment charged to customer i .

Consider a mechanism (q, t) and suppose customers report valuations $r := (r_1, \dots, r_N)$ and flexibility levels $c := (c_1, \dots, c_N)^2$. The mechanism then results in an allocation matrix $q(r, c)$ and payments $t(r, c)$. Let $a(b_i)$ be a $1 \times M$ dimensional vector whose first $|\mathcal{B}_{b_i}|$ entries are 1 and the rest are 0. In other words, the j entry of $a(b_i)$ is given as

$$a_j(b_i) = \begin{cases} 1 & \text{if } 1 \leq j \leq |\mathcal{B}_{b_i}| \\ 0 & \text{otherwise} \end{cases}. \quad (5)$$

¹ θ_i^{\min} is assumed to be non-negative.

²Customers may not report their valuations and/or flexibility levels truthfully, so r_i and c_i may be different from θ_i and b_i , respectively.

Customer i 's utility function can then be written in terms of its true valuation θ_i , true flexibility level b_i , the reported valuations r and the reported flexibility levels c as

$$u_i(\theta_i, r, b_i, c) = \theta_i a(b_i) q_i^T(r, c) - t_i(r, c), \quad (6)$$

where $q_i(r, c)$ is the i^{th} row of the allocation matrix $q(r, c)$.

The auctioneer's objective is to find a mechanism that maximizes its expected revenue while satisfying *Bayesian Incentive Compatibility* and *Individual Rationality* constraints. We describe these constraints below.

In a Bayesian incentive compatible (BIC) mechanism, truthful reporting of private information (valuations and flexibility levels in our setup) constitutes an equilibrium of the Bayesian game induced by the mechanism. In other words, each customer would prefer to report its true valuation and flexibility level provided that all other customers have adopted truth-telling strategy. Bayesian incentive compatibility can be described by the following constraint:

$$\mathbb{E}_{\theta_{-i}, b_{-i}} \left[\theta_i a(b_i) q_i^T(\theta, b) - t_i(\theta, b) \right] \geq \\ \mathbb{E}_{\theta_{-i}, b_{-i}} \left[\theta_i a(b_i) q_i^T(r_i, \theta_{-i}, c_i, b_{-i}) - t_i(r_i, \theta_{-i}, c_i, b_{-i}) \right], \\ \forall \theta_i, r_i \in \Theta_i, c_i, b_i \in \{1, 2, \dots, k\}, \forall i \in \mathcal{N}. \quad (7)$$

(7) states that customer i with type (θ_i, b_i) has a better expected utility if it reports its type truthfully as compared to its utility if it reports some other type (r_i, c_i) .

Individual Rationality (IR) constraint implies that the customer's expected utility at the truthful reporting equilibrium is non-negative. This can be expressed as:

$$\mathbb{E}_{\theta_{-i}, b_{-i}} \left[\theta_i a(b_i) q_i^T(\theta, b) - t_i(\theta, b) \right] \geq 0, \\ \forall \theta_i \in \Theta_i, b_i \in \{1, 2, \dots, k\}, \forall i \in \mathcal{N}. \quad (8)$$

The expected revenue under a BIC and IR mechanism is $\mathbb{E}_{\theta, b} \left\{ \sum_{i=1}^N t_i(\theta, b) \right\}$ when all customers adopt the truthful strategy. The auction design problem can now be formulated as

$$\max_{(q, t)} \mathbb{E}_{\theta, b} \left\{ \sum_{i=1}^N t_i(\theta, b) \right\}, \text{ subject to (7), (8).}$$

We make two assumptions for the auction design problem. Firstly, we assume that the allocation rule q does not give a customer any good that is outside its *reported* flexibility set. This can be formalized as follows:

Assumption 1. We assume that for each $i \in \mathcal{N}$, $q_i(r, c)$ can have non-zero entries only in its first $|\mathcal{B}_{c_i}|$ positions. (Recall that the first $|\mathcal{B}_{c_i}|$ positions of $q_i(\cdot, \cdot)$ correspond to goods in \mathcal{B}_{c_i} .)

The above assumption simply means that the mechanism respects the customer's reported flexibility constraint. We further assume that customers cannot over-report their flexibility level:

Assumption 2. For each $i \in \mathcal{N}$, customer i 's reported flexibility level c_i cannot exceed its true flexibility level b_i .

The above assumption can be justified by noting that customers gain no utility from getting a good outside their true

flexibility set and may in fact suffer a significant disutility if allocated a good outside their true flexibility set. To avoid the risk of getting an unusable or damaging good, customers may reasonably restrict themselves to under-reporting or truthfully reporting their flexibility level. Assumption 2 implies that the BIC constraint in (7) need not consider the case of $c_i > b_i$.

A. Examples

1. Consider the case where $\mathcal{N} = \{1, 2\}$, $\mathcal{M} = \{1, 2\}$, that is, there are two customers and two goods. Let $\mathcal{B}_1 = \{1\}$ and $\mathcal{B}_2 = \{1, 2\}$. Customer 1's type is $(\theta_1 = 1, b_1 = 2)$ with probability 1. Customer 2's type (θ_2, b_2) is uniformly distributed over the set $[0.5, 2] \times \{1, 2\}$. Consider a mechanism for this case that operates as follows:

- (i) Each customer reports a valuation and a flexibility level.
- (ii) If customer i has the highest valuation (assume that ties are resolved randomly), the mechanism allocates a good to customer i from its reported flexibility set and charges it the second highest reported valuation.
- (iii) The other customer is allocated a good from its flexibility set if such a good is available and it is charged a reserve price of 0.5.

Suppose that customer 1 reports its type truthfully and that customer 2's true type is $(\theta_2 = 2, b_2 = 2)$. If customer 2 also reports its type truthfully, it will obtain a good at a price of 1 (the second highest reported valuation) resulting in a utility of $2 - 1 = 1$. On the other hand, if it misreports its type as $(0.5, 2)$, it will obtain a good at a price of 0.5 resulting in a utility of 1.5.

2. Consider the same setup as above but with the following mechanism:

- (i) Each customer reports a valuation and a flexibility level.
- (ii) If customer i has the highest valuation (assume that ties are resolved randomly), the mechanism allocates a good to customer i from its reported flexibility set. Customer i is charged the reported valuation of the other customer *if the two reported the same flexibility level*, otherwise it pays a reserve price of 0.5.
- (iii) The other customer is allocated a good from its flexibility set if such a good is available and it is charged a reserve price of 0.5.

Suppose that customer 1 reports its type truthfully and that customer 2's true type is $(\theta_2 = 2, b_2 = 2)$. If customer 2 also reports its type truthfully, it will obtain a good at a price of 1 resulting in a utility of $2 - 1 = 1$. On the other hand, if it misreports its type as $(2, 1)$, it will obtain a good at a price of 0.5 resulting in a utility of 1.5.

Thus, in both the examples above, the mechanism described is not incentive compatible.

III. CHARACTERIZATION OF BIC AND IR MECHANISMS

Suppose all customers other than i report their valuations and flexibility levels truthfully. We can then define customer i 's

expected allocation and payment under the mechanism (q, t) when it reports $r_i \in \Theta_i$, $c_i \in \{1, 2, \dots, k\}$ as:

$$Q_i(r_i, c_i) := \mathbb{E}_{\theta_{-i}, b_{-i}} [q_i(r_i, \theta_{-i}, c_i, b_{-i})], \quad (9)$$

$$T_i(r_i, c_i) := \mathbb{E}_{\theta_{-i}, b_{-i}} [t_i(r_i, \theta_{-i}, c_i, b_{-i})]. \quad (10)$$

We can now rewrite equations (7) and (8) in terms of the interim quantities defined in (9)-(10). The BIC constraint for misreports of valuations and flexibility levels becomes:

$$\begin{aligned} \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) &\geq \\ \theta_i a(b_i) Q_i^T(r_i, c_i) - T_i(r_i, c_i), & \\ \forall \theta_i, r_i \in \Theta_i, c_i \leq b_i, c_i, b_i \in \{1, 2, \dots, k\}, \forall i \in \mathcal{N}. & \end{aligned} \quad (11)$$

The IR constraint is rewritten as:

$$\begin{aligned} \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) &\geq 0, \\ \forall \theta_i \in \Theta_i, b_i \in \{1, 2, \dots, k\}, \forall i \in \mathcal{N}. & \end{aligned} \quad (12)$$

The BIC constraint in (11) captures all possible ways that a customer may misreport its private information. It includes the following two special sub-classes of constraints:

1) BIC constraint for misreporting only valuation:

$$\begin{aligned} \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) &\geq \\ \theta_i a(b_i) Q_i^T(r_i, b_i) - T_i(r_i, b_i), & \\ \forall \theta_i, r_i \in \Theta_i, \forall b_i \in \{1, 2, \dots, k\}, \forall i \in \mathcal{N}. & \end{aligned} \quad (13)$$

2) BIC constraint for misreporting only flexibility level:

$$\begin{aligned} \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) &\geq \\ \theta_i a(b_i) Q_i^T(\theta_i, c_i) - T_i(\theta_i, c_i), & \\ \forall \theta_i \in \Theta_i, c_i \leq b_i, c_i, b_i \in \{1, 2, \dots, k\}, \forall i \in \mathcal{N}. & \end{aligned} \quad (14)$$

The following result relates the above constraints for “one-dimensional” misreports to the general BIC constraint in (11).

Lemma 1. *The BIC constraint for misreporting both valuation and flexibility level implies and is implied by the BIC constraints for misreporting only valuation and misreporting only flexibility level. That is, (11) holds if and only if (13) and (14) hold.*

Proof. See Appendix A. □

Lemma 1 allows us to replace the general BIC constraint for two-dimensional misreports by the simpler one-dimensional BIC constraints given in (13) and (14). The auction design problem now becomes:

$$\max_{(q, t)} \mathbb{E}_{\theta, b} \left\{ \sum_{i=1}^N t_i(\theta, b) \right\}, \text{ subject to (12), (13), (14).}$$

We will now derive alternative characterizations of the constraints (12), (13), (14) that will be helpful for finding the optimal mechanism.

Lemma 2. *A mechanism (q, t) satisfies the BIC constraint for misreporting only valuation (as given in (13)) if and only if for all $i \in \mathcal{N}$, $a(b_i) Q_i^T(r_i, b_i)$ is non-decreasing in r_i and*

$$T_i(r_i, b_i) = K_i(b_i) + r_i a(b_i) Q_i^T(r_i, b_i) - a(b_i) \int_{\theta_i^{\min}}^{r_i} Q_i^T(s, b_i) ds, \quad (15)$$

for all b_i .

Proof. See Appendix B. \square

Lemma 3. Suppose the mechanism (q, t) satisfies the BIC constraint for misreporting only valuation (as given in (13)). Then, it satisfies the IR constraint (12) if and only if

$$\theta_i^{\min} a(b_i) Q_i^T(\theta_i^{\min}, b_i) - T_i(\theta_i^{\min}, b_i) \geq 0. \quad (16)$$

Proof. Clearly (12) implies (16). The converse follows from Lemma 2 by noting that

$$K_i(b_i) = T_i(\theta_i^{\min}, b_i) - \theta_i^{\min} a(b_i) Q_i^T(\theta_i^{\min}, b_i),$$

and that the right hand side above is non-positive due to (16). \square

Using the above two lemmas, we derive a sufficient condition for the mechanism to satisfy the BIC constraint for misreporting only flexibility level.

Lemma 4. Suppose the mechanism (q, t) is individually rational and satisfies the BIC constraint for misreporting only valuation (as given in (13)). Then the mechanism (q, t) satisfies the BIC constraint for misreporting only flexibility level if the following are true:

- (i) $a(c_i) Q_i^T(\theta_i, c_i)$ is non-decreasing in c_i , $\forall \theta_i \in \Theta_i, \forall i \in \mathcal{N}$, and
- (ii) $T_i(\theta_i^{\min}, c_i) = 0$, $\forall c_i \in \{1, 2, \dots, k\}$, $\forall i \in \mathcal{N}$.

Proof. See Appendix C. \square

IV. REVENUE MAXIMIZING MECHANISM

We can now use the results of Section III to simplify the objective of the auction design problem. The total expected revenue can be written as

$$\begin{aligned} \mathbb{E}_{\theta, b} \left\{ \sum_{i=1}^N t_i(\theta, b) \right\} &= \sum_{i=1}^N \mathbb{E}_{\theta_i, b_i} \left[\mathbb{E}_{\theta_{-i}, b_{-i}} [t_i(\theta_i, \theta_{-i}, b_i, b_{-i})] \right] \\ &= \sum_{i=1}^N \mathbb{E}_{\theta_i, b_i} [T_i(\theta_i, b_i)]. \end{aligned} \quad (17)$$

For a mechanism that is individually rational and Bayesian incentive compatible, we can use the result in Lemma 2 to plug in the expression for $T_i(\theta_i, b_i)$. After some simplifications we obtain that

$$\begin{aligned} \mathbb{E}_{\theta_i, b_i} [T_i(\theta_i, b_i)] &= \mathbb{E}_{b_i} [K_i(b_i)] \\ &+ \sum_b \int_{\theta} \left[a(b_i) q_i^T(\theta, b) \left(\theta_i - \frac{1 - F_{\theta|B}(\theta_i|b_i)}{f_{\theta|B}(\theta_i|b_i)} \right) \right] f(\theta, b) d\theta. \end{aligned} \quad (18)$$

where, $f_{\theta|B}(\theta_i|b_i)$ is the probability density function of customer i 's valuation conditioned on its flexibility level b_i and $F_{\theta|B}(\theta_i|b_i)$ is the corresponding cumulative distribution

function. The term $\left(\theta_i - \frac{1 - F_{\theta|B}(\theta_i|b_i)}{f_{\theta|B}(\theta_i|b_i)} \right)$ is referred to as the customer's *virtual type or virtual valuation* in economics terminology and we denote it by $w_i(\theta_i, b_i)$.

We can now rewrite the auctioneer's total expected revenue in (17) as:

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}_{\theta_i, b_i} [T_i(\theta_i, b_i)] &= \sum_{i=1}^N \mathbb{E}_{b_i} [K_i(b_i)] \\ &+ \sum_b \int_{\theta} \sum_{i=1}^N \left[a(b_i) q_i^T(\theta, b) w_i(\theta_i, b_i) \right] f(\theta, b) d\theta. \end{aligned} \quad (19)$$

The second term on the right hand side in (19) is completely determined by the choice of the allocation function $q(\cdot, \cdot)$. Also, note that Lemmas 2 and 3 imply that $K_i(b_i) = T_i(\theta_i^{\min}, b_i) - \theta_i^{\min} a(b_i) Q_i^T(\theta_i^{\min}, b_i) \leq 0$. Therefore, a mechanism (q, t) that maximizes the second term on the right hand side in (19) and ensures that $K_i(b_i) = 0$ for all i and b_i while satisfying the BIC and IR constraints would provide the largest expected revenue.

In order to simplify the maximization of the second term in (19) we assume that the virtual types $\left(\theta_i - \frac{1 - F_{\theta|B}(\theta_i|b_i)}{f_{\theta|B}(\theta_i|b_i)} \right)$ are non-decreasing in θ_i and b_i . Such a condition holds if $\frac{f_{\theta|B}(\theta_i|b_i)}{1 - F_{\theta|B}(\theta_i|b_i)}$ is non-decreasing in θ_i and b_i . This condition can be viewed as a generalization of the increasing hazard rate condition [22, Chapter 2] and is similar to the condition about monotonicity of virtual valuations described in [23] for multidimensional private types. We formally state this condition and our assumptions below.

Generalized Monotone Hazard Rate Condition: The type (θ_i, b_i) is said to be partially ordered above (θ'_i, b'_i) , and this relation denoted by $(\theta_i, b_i) \succeq (\theta'_i, b'_i)$, if $\theta_i \geq \theta'_i$ and $b_i \geq b'_i$. The distribution $f_i(\cdot, \cdot)$ then satisfies the generalized monotone hazard rate condition if:

$$(\theta_i, b_i) \succeq (\theta'_i, b'_i) \implies \frac{f_{\theta|B}(\theta_i|b_i)}{1 - F_{\theta|B}(\theta_i|b_i)} \geq \frac{f_{\theta|B}(\theta'_i|b'_i)}{1 - F_{\theta|B}(\theta'_i|b'_i)}; \quad (20)$$

Further, $b_i > b'_i$ and $\theta_i \geq \theta'_i$ imply

$$\frac{f_{\theta|B}(\theta_i|b_i)}{1 - F_{\theta|B}(\theta_i|b_i)} > \frac{f_{\theta|B}(\theta'_i|b'_i)}{1 - F_{\theta|B}(\theta'_i|b'_i)}. \quad (21)$$

Assumption 3. We assume that the probability density functions $f_i(\cdot, \cdot)$ satisfy the generalized monotone hazard rate condition for all $i \in \mathcal{N}$.

Assumption 4. We assume that $w_i(\theta_i^{\min}, b_i) < 0$, $\forall b_i \in \{1, 2, \dots, k\}$, $\forall i \in \mathcal{N}$.

The following theorem characterizes the optimal mechanism under the above assumptions.

Theorem 1. Consider the allocation and tax functions (q^*, t^*) defined below

$$q^*(\theta, b) \in \operatorname{argmax}_{\mathbf{A} \in \mathcal{S}} \sum_{i=1}^N (a(b_i) \mathbf{A}_i^T) w_i(\theta_i, b_i), \quad (22)$$

where \mathbf{A}_i is the i th row of matrix \mathbf{A} ;

$$t_i^*(\theta, b) := \theta_i a(b_i) q_i^{*T}(\theta, b) - a(b_i) \int_{\theta_i^{min}}^{\theta_i} q_i^{*T}(s, \theta_{-i}, b) ds. \quad (23)$$

Then, under Assumptions 1-4, (q^*, t^*) is a revenue-maximizing Bayesian incentive compatible and individually rational mechanism.

Proof. See Appendix D. \square

The optimal allocation matrix $q^*(\theta, b)$ given in (22) is the solution of an integer program and hence computationally hard to obtain. Moreover, each type profile $(\theta, b) \in \Theta \times \{1, 2, \dots, k\}^N$ requires the solution of a different integer program. Similarly, the characterization of payments given by (23) is not very useful from a computational viewpoint as it requires the solution of a continuum of integer programs. In the next section, we leverage the nested structure imposed on customers' flexibility sets to simplify the optimal mechanism.

V. A CANDIDATE REVENUE MAXIMIZING MECHANISM

Based on their true flexibility sets, we can divide the customers into k classes: \mathcal{C}_l is the set of customers with flexibility set \mathcal{B}_l . Clearly, $\mathcal{N} = \bigcup_{l=1}^k \mathcal{C}_l$ and for $i \neq j$, $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$. We define

$$\begin{aligned} n_l &:= |\mathcal{C}_l|, \quad l = 1, \dots, k, \\ m_l &:= |\mathcal{B}_l \setminus \mathcal{B}_{l-1}|, \quad l = 2, \dots, k, \quad m_1 := |\mathcal{B}_1|. \end{aligned} \quad (24)$$

We also define the vectors \mathbf{n} and \mathbf{m} as

$$\mathbf{n} := (n_1, n_2, \dots, n_k), \quad \mathbf{m} := (m_1, m_2, \dots, m_k). \quad (25)$$

The vector \mathbf{n} is referred to as the *demand profile* and the vector \mathbf{m} is referred to as the *supply profile*.

A. Supply Adequacy Problem

Before describing the optimal mechanism, we will need to answer two questions:

- 1) Given a supply profile \mathbf{m} and a demand profile \mathbf{n} , can the available goods be used to satisfy all customers? In other words, does there exist an allocation matrix $\mathbf{A} \in \{0, 1\}^{N \times M}$ such that

$$\begin{aligned} \sum_{j \in \phi_i} \mathbf{A}(i, j) &= 1, \quad \forall i \in \mathcal{N}, \\ \sum_{i=1}^N \mathbf{A}(i, j) &\leq 1, \quad \forall j \in \mathcal{M}. \end{aligned} \quad (26)$$

The above conditions on \mathbf{A} ensure that each customer gets a good from its flexibility set and that a good is not allocated to multiple customers. If such an allocation matrix exists, we say that the supply profile \mathbf{m} is adequate for the demand profile \mathbf{n} .

- 2) If the supply profile \mathbf{m} is not adequate for the demand profile \mathbf{n} , we have to remove some customers from

the demand profile to achieve adequacy. What is the minimum number of customers that must be removed to achieve adequacy?

Borrowing ideas from [24], we provide answers to the above questions in Lemmas 5 and 6 below.

Lemma 5. We say that $\mathbf{n} \prec_w \mathbf{m}$ if the following k inequalities hold:

$$\sum_{i=1}^l n_i \leq \sum_{i=1}^l m_i, \quad l = 1, 2, \dots, k. \quad (27)$$

The supply profile \mathbf{m} is adequate for the demand profile \mathbf{n} if and only if $\mathbf{n} \prec_w \mathbf{m}$.

Proof. See Appendix E. \square

If the supply profile \mathbf{m} is not adequate, we have to remove some customers from the demand profile. Consider a demand profile $\tilde{\mathbf{n}} \leq \mathbf{n}$ obtained by removing some customers. This new demand profile will result in adequacy if and only if $\tilde{\mathbf{n}} \prec_w \mathbf{m}$. Thus, the minimum number of customers to be removed to achieve adequacy is given by the following optimization problem:

$$\min_{\tilde{\mathbf{n}}} \sum_{i=1}^k (n_i - \tilde{n}_i), \quad \text{subject to } \tilde{\mathbf{n}} \prec_w \mathbf{m}, \quad \tilde{\mathbf{n}} \leq \mathbf{n}. \quad (28)$$

$\tilde{\mathbf{n}}$ in the above optimization problem is a vector of non-negative integers. This integer program has a simple solution described in the following lemma.

Lemma 6. Define $r_1^* := (n_1 - m_1)^+$. For $2 \leq j \leq k$, recursively define r_j^* as the solution of the following one-dimensional integer program:

$$\begin{aligned} r_j^* &= \operatorname{argmin}_{r_j \in \mathbb{Z}^+} r_j \\ \text{subject to } \sum_{l=1}^{j-1} (n_l - r_l^*) + (n_j - r_j) &\leq \sum_{l=1}^j m_l. \end{aligned} \quad (29)$$

Then, (i) for $j = 1, \dots, k$, at least $\sum_{i=1}^j r_i^*$ customers must be removed from the first j classes to satisfy the inequalities (27) of Lemma 5; (ii) $\sum_{i=1}^k r_i^*$ is the minimum value of the integer program in (28).

Proof. See Appendix F. \square

B. Optimal Allocation

We can now use the results of Section V-A to find the optimal allocation for a given type profile (θ, b) . Recall from Theorem 1 that the optimal allocation is given as

$$q^*(\theta, b) \in \operatorname{argmax}_{\mathbf{A} \in \mathcal{S}} \sum_{i=1}^N (a(b_i) \mathbf{A}_i^T) w_i(\theta_i, b_i).$$

We describe the optimal allocation in the following steps:

- 1) Firstly, any customer l with $w_l(\theta_l, b_l) \leq 0$ is immediately removed from consideration (that is, it is not allocated any good). Since virtual valuation is a non-decreasing function of true valuation, $w_l(\theta_l, b_l) \leq 0$ if and only if $\theta_l \leq \theta_{l, b_l}^{res}$, where θ_{l, b_l}^{res} is a threshold based on the

probability distribution of θ_l conditioned on the flexibility level b_l . This threshold is called the reserve price for customer l with flexibility level b_l .

For each class of customers, we define the subset of customers who have positive virtual valuations:

$$\mathcal{C}_i^+ := \{l \in \mathcal{C}_i : w_l(\theta_l, b_l) > 0\}. \quad (30)$$

Let $n_i^+ = |\mathcal{C}_i^+|$. Define r_1^*, \dots, r_k^* as in Lemma 6 by replacing n_i with n_i^+ for all i .

- 2) Let $\mathcal{L}_1 := \mathcal{C}_1^+$. From \mathcal{L}_1 , r_1^* customers with the lowest virtual valuations are removed from consideration³. The set of remaining customers in \mathcal{L}_1 is denoted by \mathcal{N}_1 .
- 3) We now proceed iteratively: For $2 \leq i \leq k$, given the set \mathcal{N}_{i-1} , define $\mathcal{L}_i := \mathcal{N}_{i-1} \cup \mathcal{C}_i^+$. Remove r_i^* customers with lowest virtual valuations from \mathcal{L}_i . The set of remaining customers in \mathcal{L}_i is now defined as \mathcal{N}_i .
- 4) After the k^{th} iteration, all customers in \mathcal{N}_k are allocated a good from their respective flexibility sets.

The optimality of the above allocation can be intuitively explained as follows: Firstly, it is clear that an optimal allocation should not give a good to customers with non-positive virtual valuations. Among the remaining customers of class \mathcal{C}_1 , at least r_1^* customers cannot be served (see Lemma 6 with n_i replaced by n_i^+ for all i). It is easy to see that the r_1^* customers with the lowest virtual valuations should be removed. This argument can be used iteratively. At the i th iteration, at least r_i^* additional customers need to be removed from the first i classes otherwise the i th adequacy inequality would be violated. An optimal allocation should remove r_i^* customers with lowest virtual valuations. After the k th iteration, exactly $\sum_{i=1}^k r_i^*$ customers have been removed and the remaining customers' demand profile satisfies all the adequacy inequalities.

The above optimal allocation procedure can also be described using thresholds. Define

$$w_i^{\text{thr}} := (r_i^*)^{\text{th}} \text{ lowest virtual valuation in } \mathcal{L}_i, i = 1, 2, \dots, k. \quad (31)$$

(If $r_i^* = 0$, $w_i^{\text{thr}} = 0$.)

Then, at iteration i , customers that have virtual valuations less than or equal to w_i^{thr} will be removed from the set \mathcal{L}_i . Under the optimal allocation, customer l in class \mathcal{C}_i gets a desired good if its virtual valuation exceeds 0 and thresholds $w_i^{\text{thr}}, w_{i+1}^{\text{thr}}, \dots, w_k^{\text{thr}}$. Let us define

$$\theta_{l,i}^{\text{thr}} = \left\{ x : w_l(x, i) = \max\{0, w_i^{\text{thr}}, w_{i+1}^{\text{thr}}, \dots, w_k^{\text{thr}}\} \right\}. \quad (32)$$

Because of the monotonicity of virtual valuation as a function of true valuation, customer l in class \mathcal{C}_i gets a good if $\theta_l > \theta_{l,i}^{\text{thr}}$. Thus,

$$a(i)q_l^{*T}(\theta, b) = \begin{cases} 1 & \text{if } \theta_l > \theta_{l,i}^{\text{thr}} \\ 0 & \text{otherwise} \end{cases}, \quad (33)$$

$\forall l \in \mathcal{C}_i, \forall i = 1, 2, \dots, k.$

³Ties are resolved randomly. For continuous valuations, ties happen with zero probability and therefore the allocation rule for ties does not affect expected revenue.

C. Payment Functions

We can now use the optimal allocation rule described in section V-B to simplify customers' payment functions. From (23) the optimal payment function for customer l in flexibility class \mathcal{C}_i has the following form:

$$t_l^*(\theta, b) = \theta_l a(i)q_l^{*T}(\theta, b) - a(i) \int_{\theta_l^{\min}}^{\theta_l} q_l^{*T}(s, \theta_{-l}, b) ds. \quad (34)$$

Using the definition of $a(i)q_l^{*T}(\theta, b)$ given in (33), $t_l(\theta, b)$ can be simplified as:

- 1) If $\theta_l > \theta_{l,i}^{\text{thr}}$,

$$\begin{aligned} t_l^*(\theta, b) &= \theta_l - \int_{\theta_l^{\min}}^{\theta_{l,i}^{\text{thr}}} \underbrace{a(i)q_l^{*T}(s, \theta_{-l}, b)}_{=0} ds \\ &\quad - \int_{\theta_{l,i}^{\text{thr}}}^{\theta_l} \underbrace{a(i)q_l^{*T}(s, \theta_{-l}, b)}_{=1} ds = \theta_{l,i}^{\text{thr}}. \end{aligned} \quad (35)$$

- 2) If $\theta_l \leq \theta_{l,i}^{\text{thr}}$, $t_l^*(\theta, b) = 0$.

The optimal allocation decisions and payments can thus be computed through the straightforward threshold-based procedure constructed in sections V-B and V-C. By using the nested structure of the flexibility sets, this procedure obviates the need to solve the computationally hard integer program formulated in Theorem 1.

Remark 1. Suppose that θ_l and b_l are independent random variables for all $l \in \mathcal{N}$. In this case the virtual valuation for customer l will take the following form

$$w_l(\theta_l) = \theta_l - \frac{1 - F_l(\theta_l)}{f_l(\theta_l)}, \quad \forall l \in \mathcal{N}, \quad (36)$$

If we further assume that θ_l is distributed over the set $[\theta^{\min}, \theta^{\max}]$ according to the same probability density function f for all $l \in \mathcal{N}$, then the thresholds $\theta_{l,i}^{\text{thr}}$ in (32) do not depend on l :

$$\theta_i^{\text{thr}} = \left\{ x : w(x) = \max\{0, w_i^{\text{thr}}, w_{i+1}^{\text{thr}}, \dots, w_k^{\text{thr}}\} \right\}. \quad (37)$$

Moreover, we have $\theta_1^{\text{thr}} \geq \theta_2^{\text{thr}} \geq \dots \geq \theta_k^{\text{thr}}$. The allocation and payment functions can be simplified as follows: Customer l in class \mathcal{C}_i gets a good if $\theta_l > \theta_i^{\text{thr}}$ and its payment simplifies to: $\theta_i^{\text{thr}} \mathbb{1}_{\{\theta_l > \theta_i^{\text{thr}}\}}$. It is evident in this case that more flexible customers pay less for the good than less flexible customers.

Remark 2. Suppose the probability distributions for customers' types are such that the flexibility levels are degenerate random variables. This essentially implies that the customers' flexibility sets are common knowledge. If we further assume that customers' valuations given their flexibility level are identically distributed, the same observation as in Remark 1 follows: Customer l in class \mathcal{C}_i gets a good if $\theta_l > \theta_i^{\text{thr}}$ and its payment simplifies to: $\theta_i^{\text{thr}} \mathbb{1}_{\{\theta_l > \theta_i^{\text{thr}}\}}$.

VI. CONCLUSION

We studied the problem of designing revenue-maximizing auctions for allocating multiple goods to flexible customers. In our model, each customer is interested in a subset of goods known as its flexibility set and wants to consume one good from this set. A customer's flexibility set and its utility from consuming a good from its flexibility set are its private information. We characterized the allocation rule for an incentive compatible, individually rational and revenue-maximizing auction as the solution to an integer program. The corresponding payment rule was described by an integral equation. We then leveraged the nestedness of flexibility sets to simplify the optimal auction and provided a complete characterization of allocations and payments in terms of simple thresholds.

It would be interesting to study this auction problem under dynamic settings where the set of customers and/or goods can change over time. In such a setting customers may have richer private information that includes their valuation, flexibility sets as well as their temporal presence information. Moreover, dynamic models can incorporate supply uncertainties to capture scenarios where the seller relies on uncertain and time-varying resources (such as renewable energy) to serve its customers. The auction mechanism then needs to make sequential decisions based on information revealed at or before the current time. Investigating these dynamic mechanism design problems will be a key task for future research.

APPENDIX A PROOF OF LEMMA 1

Clearly (11) implies (13) and (14). To prove the converse, consider flexibility levels c_i and b_i with $c_i \leq b_i$. From (14), we have

$$\begin{aligned} & \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) \\ & \geq \theta_i a(b_i) Q_i^T(\theta_i, c_i) - T_i(\theta_i, c_i). \end{aligned} \quad (38)$$

Consider $\theta_i, r_i \in \Theta_i$. From (13) we have

$$\begin{aligned} & \theta_i a(c_i) Q_i^T(\theta_i, c_i) - T_i(\theta_i, c_i) \\ & \geq \theta_i a(c_i) Q_i^T(r_i, c_i) - T_i(r_i, c_i). \end{aligned} \quad (39)$$

Adding the inequalities in (38) and (39) we obtain:

$$\begin{aligned} & \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) + \theta_i a(c_i) Q_i^T(\theta_i, c_i) \geq \\ & \theta_i a(c_i) Q_i^T(r_i, c_i) - T_i(r_i, c_i) + \theta_i a(b_i) Q_i^T(\theta_i, c_i). \end{aligned} \quad (40)$$

Because of Assumption 1 we have $a(c_i)Q_i^T(\theta_i, c_i) = a(b_i)Q_i^T(\theta_i, c_i)$ and $a(c_i)Q_i^T(r_i, c_i) = a(b_i)Q_i^T(r_i, c_i)$. (40) can then be written as

$$\begin{aligned} & \theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) \geq \\ & \theta_i a(b_i) Q_i^T(r_i, c_i) - T_i(r_i, c_i), \end{aligned} \quad (41)$$

which is the two-dimensional BIC constraint of (11). This concludes the proof.

APPENDIX B PROOF OF LEMMA 2

Sufficiency: Suppose $a(b_i)Q_i^T(r_i, b_i)$ is non-decreasing in r_i and customer i 's expected payment is of the form given in (15). Suppose customer i 's true type is (θ_i, b_i) and it reports (r_i, b_i) . Its expected utility is:

$$U_i(\theta_i, r_i, b_i, b_i) = \theta_i a(b_i) Q_i^T(r_i, b_i) - T_i(r_i, b_i). \quad (42)$$

We can then use (15) to rewrite customer i 's expected utility as:

$$\begin{aligned} U_i(\theta_i, r_i, b_i, b_i) &= (\theta_i - r_i) a(b_i) Q_i^T(r_i, b_i) \\ &+ a(b_i) \int_{\theta_i^{min}}^{r_i} Q_i^T(s, b_i) ds - K_i(b_i). \end{aligned} \quad (43)$$

We now need to show that $U_i(\theta_i, \theta_i, b_i, b_i) \geq U_i(\theta_i, r_i, b_i, b_i)$, $\theta_i, r_i \in \Theta_i$, $b_i \in \{1, 2, \dots, k\}$, to conclude Bayesian incentive compatibility in valuation for customer i . We use the form given in (15) to write

$$\begin{aligned} & U_i(\theta_i, \theta_i, b_i, b_i) - U_i(\theta_i, r_i, b_i, b_i) \\ &= a(b_i) \int_{\theta_i^{min}}^{\theta_i} Q_i^T(s, b_i) ds - a(b_i) \int_{\theta_i^{min}}^{r_i} Q_i^T(s, b_i) ds \\ &+ (r_i - \theta_i) a(b_i) Q_i^T(r_i, b_i) \\ &= (r_i - \theta_i) a(b_i) Q_i^T(r_i, b_i) + a(b_i) \int_{r_i}^{\theta_i} Q_i^T(s, b_i) ds \\ &= \int_{r_i}^{\theta_i} a(b_i) \{Q_i^T(s, b_i) - Q_i^T(r_i, b_i)\} ds. \end{aligned} \quad (44)$$

It is straightforward to verify that because of $a(b_i)Q_i^T(r_i, b_i)$ being non-decreasing in r_i , the expression in (44) is non-negative for both $r_i < \theta_i$ and $r_i > \theta_i$. Hence

$$U_i(\theta_i, \theta_i, b_i, b_i) \geq U_i(\theta_i, r_i, b_i, b_i) \text{ for } \theta_i, r_i \in \Theta_i, \quad (45)$$

which establishes Bayesian incentive compatibility of the mechanism (q, t) in valuation for customer i .

Necessity: Suppose (q, t) is Bayesian incentive compatible in valuation. Consider two candidate valuations $x, y \in \Theta_i$, $x < y$ that customer i might have. First, assume (x, b_i) is customer i 's true type. Then BIC in valuation implies

$$xa(b_i)Q_i^T(x, b_i) - T_i(x, b_i) \geq xa(b_i)Q_i^T(y, b_i) - T_i(y, b_i). \quad (46)$$

Now, consider (y, b_i) to be the true type. BIC in valuation gives

$$ya(b_i)Q_i^T(y, b_i) - T_i(y, b_i) \geq ya(b_i)Q_i^T(x, b_i) - T_i(x, b_i). \quad (47)$$

Adding (46) and (47) and simplifying gives

$$a(b_i)Q_i^T(y, b_i) \geq a(b_i)Q_i^T(x, b_i). \quad (48)$$

Therefore, $a(b_i)Q_i^T(r_i, b_i)$ is non-decreasing in r_i .

Let us define $V_i(\theta_i, b_i)$ as customer i 's expected utility when its valuation is θ_i and its flexibility level is b_i and it adopts truth-telling strategy:

$$V_i(\theta_i, b_i) := U_i(\theta_i, \theta_i, b_i, b_i). \quad (49)$$

Using Bayesian incentive compatibility in valuation (49) can be written as

$$\begin{aligned} V_i(\theta_i, b_i) &= \max_{r_i \in \Theta_i} U_i(\theta_i, r_i, b_i, b_i) \\ &= \max_{r_i \in \Theta_i} \theta_i a(b_i) Q_i^T(r_i, b_i) - T_i(r_i, b_i). \end{aligned} \quad (50)$$

Using the integral form of the Envelope Theorem as stated in Theorem 3.1 in [25, Chapter 3] and (50) it follows that $V_i(\theta_i, b_i)$ satisfies the following equation:

$$V_i(\theta_i, b_i) = V_i(\theta_i^{\min}, b_i) + \int_{\theta_i^{\min}}^{\theta_i} a(b_i) Q_i^T(s, b_i) ds. \quad (51)$$

Using (49) and (42) in (51), it then follows that $T_i(\theta_i, b_i)$ satisfies the following equation:

$$\begin{aligned} T_i(\theta_i, b_i) &= T_i(\theta_i^{\min}, b_i) - \theta_i^{\min} a(b_i) Q_i^T(\theta_i^{\min}, b_i) \\ &\quad + \theta_i a(b_i) Q_i^T(\theta_i, b_i) - a(b_i) \int_{\theta_i^{\min}}^{\theta_i} Q_i^T(s, b_i) ds. \end{aligned} \quad (52)$$

(52) establishes (15) with $K_i(b_i) = T_i(\theta_i^{\min}, b_i) - \theta_i^{\min} a(b_i) Q_i^T(\theta_i^{\min}, b_i)$.

APPENDIX C PROOF OF LEMMA 4

Suppose (q, t) is individually rational and Bayesian incentive compatible in valuation and satisfies conditions (i) and (ii) of Lemma 4. For a customer of true type (θ_i, b_i) who reports (θ_i, c_i) , $c_i \leq b_i$ consider

$$\theta_i a(b_i) Q_i^T(\theta_i, b_i) - T_i(\theta_i, b_i) - (\theta_i a(b_i) Q_i^T(\theta_i, c_i) - T_i(\theta_i, c_i)). \quad (53)$$

Using (15) from Lemma 2 and the second condition of Lemma 4 for the two $T_i(\cdot, \cdot)$ terms in (53), we obtain:

$$\begin{aligned} &\int_{\theta_i^{\min}}^{\theta_i} (a(b_i) Q_i^T(s, b_i) - a(c_i) Q_i^T(s, c_i)) ds \\ &+ \theta_i^{\min} (a(b_i) Q_i^T(\theta_i^{\min}, b_i) - a(c_i) Q_i^T(\theta_i^{\min}, c_i)). \end{aligned} \quad (54)$$

Since $a(b_i) Q_i^T(r_i, b_i)$ is assumed to be non-decreasing in b_i , the integral term as well as the term $(a(b_i) Q_i^T(\theta_i^{\min}, b_i) - a(c_i) Q_i^T(\theta_i^{\min}, c_i))$ are non-negative. Thus, the expression in (53) is non-negative and hence the BIC constraint in flexibility level (equation (14)) is satisfied.

APPENDIX D PROOF OF THEOREM 1

We first establish that the mechanism (q^*, t^*) is Bayesian incentive compatible and individually rational. Based on the results of Lemmas 1 - 4, it is sufficient to show the following:

- (i) Customer i 's expected payment on reporting r_i and c_i , $T_i^*(r_i, c_i)$, satisfies (15),
- (ii) $T_i^*(\theta_i^{\min}, c_i) = 0$, $\forall c_i \in \{1, 2, \dots, k\}$,
- (iii) The expected allocation, $a(c_i) Q_i^{*T}(r_i, c_i)$, is non-decreasing in r_i and c_i .

By taking the expectation of $t_i^*(\theta, b)$ over (θ_{-i}, b_{-i}) in (23), it is easily established that the expected payment satisfies (15) with $K_i(b_i) = 0$. Furthermore, since Assumption 4 states that $w_i(\theta_i^{\min}, b_i) < 0$, it follows that $a(b_i) q_i^{*T}(\theta_i^{\min}, \theta_{-i}, b) = 0$. If this were not the case then, q^* could not have achieved the maximum in (22). Evaluating (23) at θ_i^{\min} then shows that $t_i^*(\theta_i^{\min}, \theta_{-i}, b) = 0$ which further implies that $T_i^*(\theta_i^{\min}, b_i) = 0$.

In order to establish monotonicity of $a(c_i) Q_i^{*T}(r_i, c_i)$ in r_i , it is sufficient to argue that $a(c_i) q_i^{*T}(r_i, \theta_{-i}, c_i, b_{-i})$ is non-decreasing in r_i . The proof is similar to the arguments in chapters 2-3 of [22] and basically follows from the fact virtual type $w_i(r_i, c_i)$ is non-decreasing in r_i .

To establish monotonicity of $a(c_i) Q_i^{*T}(r_i, c_i)$ in c_i , it suffices to show that for any two candidate flexibility levels $\gamma, \lambda \in \{1, 2, \dots, k\}$, $\gamma < \lambda$, we will have

$$a(\gamma) q_i^{*T}(\theta, \gamma, b_{-i}) \leq a(\lambda) q_i^{*T}(\theta, \lambda, b_{-i}), \quad (55)$$

for all θ and b_{-i} .

For the type profile (θ, γ, b_{-i}) , the maximum value of the objective function in (22) is $a(\gamma) q_i^{*T}(\theta, \gamma, b_{-i}) w_i(\theta_i, \gamma) + \sum_{j \neq i} a(b_j) q_j^{*T}(\theta, \gamma, b_{-i}) w_j(\theta_j, b_j)$. Therefore, we must have

$$\begin{aligned} &a(\gamma) q_i^{*T}(\theta, \gamma, b_{-i}) w_i(\theta_i, \gamma) \\ &+ \sum_{j \neq i} a(b_j) q_j^{*T}(\theta, \gamma, b_{-i}) w_j(\theta_j, b_j) \\ &\geq a(\gamma) q_i^{*T}(\theta, \lambda, b_{-i}) w_i(\theta_i, \gamma) \\ &+ \sum_{j \neq i} a(b_j) q_j^{*T}(\theta, \lambda, b_{-i}) w_j(\theta_j, b_j). \end{aligned} \quad (56)$$

Similarly, when the type profile is $(\theta, \lambda, b_{-i})$, the maximum value of the objective function in (22) is $a(\lambda) q_i^{*T}(\theta, \lambda, b_{-i}) w_i(\theta_i, \lambda) + \sum_{j \neq i} a(b_j) q_j^{*T}(\theta, \lambda, b_{-i}) w_j(\theta_j, b_j)$. Therefore, we must have

$$\begin{aligned} &a(\lambda) q_i^{*T}(\theta, \lambda, b_{-i}) w_i(\theta_i, \lambda) \\ &+ \sum_{j \neq i} a(b_j) q_j^{*T}(\theta, \lambda, b_{-i}) w_j(\theta_j, b_j) \\ &\geq a(\lambda) q_i^{*T}(\theta, \gamma, b_{-i}) w_i(\theta_i, \lambda) \\ &+ \sum_{j \neq i} a(b_j) q_j^{*T}(\theta, \gamma, b_{-i}) w_j(\theta_j, b_j). \end{aligned} \quad (57)$$

Now, adding the two inequalities (56)-(57) gives

$$\begin{aligned} &(w_i(\theta_i, \lambda) a(\lambda) - w_i(\theta_i, \gamma) a(\gamma)) q_i^{*T}(\theta, \lambda, b_{-i}) \\ &\geq (w_i(\theta_i, \lambda) a(\lambda) - w_i(\theta_i, \gamma) a(\gamma)) q_i^{*T}(\theta, \gamma, b_{-i}). \end{aligned} \quad (58)$$

Define:

$$\begin{aligned} z_1 &:= (w_i(\theta_i, \lambda)a(\lambda) - w_i(\theta_i, \gamma)a(\gamma)) q_i^{*T}(\theta, \lambda, b_{-i}), \\ z_2 &:= (w_i(\theta_i, \lambda)a(\lambda) - w_i(\theta_i, \gamma)a(\gamma)) q_i^{*T}(\theta, \gamma, b_{-i}). \end{aligned} \quad (59)$$

(58) says that

$$z_1 \geq z_2. \quad (60)$$

Let us denote $\omega_\lambda := w_i(\theta_i, \lambda)$ and $\omega_\gamma := w_i(\theta_i, \gamma)$. From the generalized monotone hazard rate condition (see (21)), we know that $\omega_\lambda > \omega_\gamma$. From the definition of vector $a(\cdot)$ (see (5)) and q^* (see Assumption 1) and the fact that $\gamma < \lambda$, it is easy to see that $a(\lambda)q_i^{*T}(\theta, \lambda, b_{-i}) \geq a(\gamma)q_i^{*T}(\theta, \lambda, b_{-i})$ and $a(\lambda)q_i^{*T}(\theta, \gamma, b_{-i}) = a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i})$.

Depending on the values of $a(\lambda)q_i^{*T}(\theta, \lambda, b_{-i})$, $a(\gamma)q_i^{*T}(\theta, \lambda, b_{-i})$ and $a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i})$, z_1 and z_2 can take the following values:

$$z_1 = \begin{cases} 0 & \text{if } a(\lambda)q_i^{*T}(\theta, \lambda, b_{-i}) = 0 \\ \omega_\lambda - \omega_\gamma & \text{if } a(\gamma)q_i^{*T}(\theta, \lambda, b_{-i}) = 1 \\ \omega_\lambda & \text{if } (a(\lambda) - a(\gamma))q_i^{*T}(\theta, \lambda, b_{-i}) = 1 \end{cases}, \quad (61)$$

$$z_2 = \begin{cases} 0 & \text{if } a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i}) = 0 \\ \omega_\lambda - \omega_\gamma & \text{if } a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i}) = 1 \end{cases}. \quad (62)$$

We can establish (55) as follows: The quantities on the left and right hand sides in (55) are either 0 or 1. If $a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i}) = 0$, then (55) is trivially true. It remains to be shown that when $a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i}) = 1$ we also have $a(\lambda)q_i^{*T}(\theta, \lambda, b_{-i}) = 1$. Suppose $a(\gamma)q_i^{*T}(\theta, \gamma, b_{-i}) = 1$ but $a(\lambda)q_i^{*T}(\theta, \lambda, b_{-i}) = 0$. This would imply that $z_2 = \omega_\lambda - \omega_\gamma$ (which is positive) and $z_1 = 0$; but then $z_1 < z_2$ which is a contradiction of (60). This proves (55).

Finally, it is straightforward to see that the allocation rule $q^*(\theta, b)$ which is defined in (22) as the maximizer of the weighted sum $\sum_{i=1}^N a(b_i)q_i^T(\theta, b)w_i(\theta_i, b_i)$, will naturally maximize the second term on the right hand side of (19). Moreover, as argued above, $K_i(b_i) = 0$ for all i and b_i under (q^*, t^*) . Thus, (q^*, t^*) maximizes the auctioneer's total expected revenue. Hence, the mechanism (q^*, t^*) is a revenue-maximizing Bayesian incentive compatible and individually rational mechanism.

APPENDIX E PROOF OF LEMMA 5

Necessity: From the adequacy condition in equation (26) we have:

$$\sum_{g \in \phi_i} \mathbf{A}(i, g) = 1, \quad \forall i \in \mathcal{N}. \quad (63)$$

Summing both sides of the above equation over the set of customers in the union of first J classes, we get:

$$\sum_{i \in \cup_{j=1}^J \mathcal{C}_j} \sum_{g \in \phi_i} \mathbf{A}(i, g) = \sum_{i \in \cup_{j=1}^J \mathcal{C}_j} 1 = \sum_{i=1}^J n_i. \quad (64)$$

The left hand side of (64) can be written as

$$\begin{aligned} \sum_{i \in \cup_{j=1}^J \mathcal{C}_j} \sum_{g \in \phi_i} \mathbf{A}(i, g) &\leq \sum_{i \in \cup_{j=1}^J \mathcal{C}_j} \sum_{g \in \mathcal{B}_J} \mathbf{A}(i, g) \\ &= \sum_{g \in \mathcal{B}_J} \sum_{i \in \cup_{j=1}^J \mathcal{C}_j} \mathbf{A}(i, g) \\ &\leq \sum_{g \in \mathcal{B}_J} \sum_{i \in \mathcal{N}} \mathbf{A}(i, g) \\ &\leq \sum_{g \in \mathcal{B}_J} 1 = \sum_{i=1}^J m_i. \end{aligned} \quad (65)$$

(64) and (65) imply that:

$$\sum_{i=1}^J n_i \leq \sum_{i=1}^J m_i, \quad J = 1, 2, \dots, k, \quad (66)$$

which proves the necessity part of the lemma.

Sufficiency: We enumerate the items in the sets $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ so that $\mathcal{B}_i = \{1, 2, \dots, \sum_{j=1}^i m_j\}$, $i = 1, 2, \dots, k$. The customers in classes $\mathcal{C}_i, i = 1, 2, \dots, k$ can be enumerated as $\mathcal{C}_1 = \{1, 2, \dots, n_1\}$ and $\mathcal{C}_i = \{1 + \sum_{j=1}^{i-1} n_j, \dots, \sum_{j=1}^i n_j\}$, $i = 2, 3, \dots, k$.

Consider an allocation where the j^{th} customer (as per the above enumeration) gets the j^{th} good (as per the above enumeration). Thus, $\mathbf{A}(i, i) = 1$, $\forall i \in \mathcal{N}$ and $\mathbf{A}(i, j) = 0$, for $j \neq i$. Since $\sum_{i=1}^l n_i \leq \sum_{i=1}^l m_i$ for $l = 1, 2, \dots, k$, one can verify that customer j will always get something from its flexibility set ϕ_j . Therefore, given the inequalities in (27), an allocation matrix can be found that satisfies the conditions in (26), which is to say that the supply profile \mathbf{m} is adequate for the demand profile \mathbf{n} .

APPENDIX F PROOF OF LEMMA 6

Consider any feasible solution of the optimization problem in (28) denoted as $(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_k)$. We will now show inductively that:

$$\sum_{j=1}^i (n_j - \tilde{n}_j) \geq \sum_{j=1}^i r_j^*, \quad \forall i = 1, 2, \dots, k. \quad (67)$$

For $i = 1$ we have:

$$\tilde{n}_1 \leq n_1, \quad \tilde{n}_1 \leq m_1 \implies \tilde{n}_1 \leq \min\{n_1, m_1\}. \quad (68)$$

From this we can write:

$$n_1 - \tilde{n}_1 \geq n_1 - \min\{n_1, m_1\} = (n_1 - m_1)^+ = r_1^*. \quad (69)$$

Now suppose the inequality in (67) holds for i . We now want to prove it also holds for $i + 1$. Let us consider two cases based on the possible values of r_{i+1}^* : 1) $r_{i+1}^* = 0$ and 2) $r_{i+1}^* > 0$. When $r_{i+1}^* = 0$, it follows directly from the induction hypothesis for i in (67) that:

$$\sum_{j=1}^{i+1} (n_j - \tilde{n}_j) \geq \sum_{j=1}^{i+1} r_j^*. \quad (70)$$

Now consider the case when $r_{i+1}^* > 0$. In this case, from the optimization constraint in (29) it can be verified that $r_{i+1}^* = n_{i+1} + \sum_{j=1}^i (n_j - r_j^*) - \sum_{j=1}^{i+1} m_j$; hence:

$$\sum_{j=1}^{i+1} (n_j - r_j^*) = \sum_{j=1}^{i+1} m_j, \quad (71)$$

which implies

$$\sum_{j=1}^{i+1} r_j^* = \sum_{j=1}^{i+1} (n_j - m_j). \quad (72)$$

From the optimization constraints in (28) we know that:

$$\sum_{j=1}^{i+1} \tilde{n}_j \leq \sum_{j=1}^{i+1} m_j. \quad (73)$$

Combining (72) and (73) we get:

$$\sum_{j=1}^{i+1} (n_j - \tilde{n}_j) \geq \sum_{j=1}^{i+1} (n_j - m_j) = \sum_{j=1}^{i+1} r_j^*. \quad (74)$$

Thus the inequality in (67) holds for $i+1$ as well. Therefore by induction we can conclude that: $\sum_{j=1}^l (n_j - \tilde{n}_j) \geq \sum_{j=1}^l r_j^*$, for $l = 1, \dots, k$. Thus, at least $\sum_{j=1}^l r_j^*$ customers that must be removed from the first l classes to satisfy the inequalities in (27) of Lemma 5.

To show that the $\sum_{j=1}^k r_j^*$ is minimum value of the integer program in (28), consider the following procedure:

- 1) Let $\mathcal{L}_1 := \mathcal{C}_1$. From \mathcal{L}_1 , r_1^* customers are removed. The set of remaining customers in \mathcal{L}_1 is denoted by \mathcal{N}_1 .
- 2) Proceed iteratively: For $2 \leq i \leq k$, given the set \mathcal{N}_{i-1} , define $\mathcal{L}_i := \mathcal{N}_{i-1} \cup \mathcal{C}_i$. Remove r_i^* customers from \mathcal{L}_i . The set of remaining customers in \mathcal{L}_i is now defined as \mathcal{N}_i .

It can be verified that the above procedure removes exactly $\sum_{j=1}^k r_j^*$ customers and creates a demand profile $\tilde{\mathbf{n}}$ that meets the adequacy condition $\tilde{\mathbf{n}} \prec_w \mathbf{m}$.

REFERENCES

- [1] E. Bitar and Y. Xu, "Deadline differentiated pricing of deferrable electric loads," *IEEE Transactions on Smart Grid*, vol. 8, no. 1, pp. 13–25, Jan 2017.
- [2] M. Khaledi and A. A. Abouzeid, "Auction-based spectrum sharing in cognitive radio networks with heterogeneous channels," in *Information Theory and Applications Workshop (ITA), 2013*. IEEE, 2013, pp. 1–8.
- [3] S. Sengupta and M. Chatterjee, "Designing auction mechanisms for dynamic spectrum access," *Mobile Networks and Applications*, vol. 13, no. 5, pp. 498–515, 2008.
- [4] Y. Zhang, D. Niyato, P. Wang, and E. Hossain, "Auction-based resource allocation in cognitive radio systems," *IEEE Communications Magazine*, vol. 50, no. 11, pp. 108–120, 2012.
- [5] M. Mangili, F. Martignon, S. Paris, and A. Capone, "Bandwidth and cache leasing in wireless information centric networks: a game theoretic study," *IEEE Transactions on Vehicular Technology*, vol. 66, no. 99, pp. 679–695, 2017.
- [6] W. Vickrey, "Counterspeculation, auctions, and competitive sealed tenders," *The Journal of finance*, vol. 16, no. 1, pp. 8–37, 1961.
- [7] E. H. Clarke, "Multipart pricing of public goods," *Public choice*, vol. 11, no. 1, pp. 17–33, 1971.
- [8] T. Groves, "Incentives in teams," *Econometrica: Journal of the Econometric Society*, pp. 617–631, 1973.
- [9] L. M. Ausubel, P. Milgrom *et al.*, "The lovely but lonely vickrey auction," *Combinatorial auctions*, vol. 17, pp. 22–26, 2006.
- [10] M. H. Rothkopf, "Thirteen reasons why the vickrey-clarke-groves process is not practical," *Operations Research*, vol. 55, no. 2, pp. 191–197, 2007.
- [11] M. Yokoo, Y. Sakurai, and S. Matsubara, "Robust combinatorial auction protocol against false-name bids," in *Game Theory and Decision Theory in Agent-Based Systems*. Springer, 2002, pp. 355–373.
- [12] R. B. Myerson, "Optimal auction design," *Mathematics of operations research*, vol. 6, no. 1, pp. 58–73, 1981.
- [13] M. Armstrong, "Optimal multi-object auctions," *Review of Economic Studies*, pp. 455–481, 2000.
- [14] J. Hartline and A. Karlin, "Profit maximization in mechanism design," in *Algorithmic Game Theory (N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, eds.)*. New York, NY, USA: Cambridge University Press, 2007, ch. 13, pp. 331–361.
- [15] P. Young and S. Zamir, *Handbook of Game Theory*. Elsevier, 2014.
- [16] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, *Algorithmic game theory*. Cambridge University Press Cambridge, 2007, vol. 1.
- [17] S. De Vries and R. V. Vohra, "Combinatorial auctions: A survey," *INFORMS Journal on computing*, vol. 15, no. 3, pp. 284–309, 2003.
- [18] P. Klemperer, "Auction theory: A guide to the literature," *Journal of economic surveys*, vol. 13, no. 3, pp. 227–286, 1999.
- [19] C. Avery and T. Hendershott, "Bundling and optimal auctions of multiple products," *The Review of Economic Studies*, vol. 67, no. 3, pp. 483–497, 2000.
- [20] J. O. Ledyard, "Optimal combinatoric auctions with single-minded bidders," in *Proceedings of the 8th ACM conference on Electronic commerce*. ACM, 2007, pp. 237–242.
- [21] V. Abhishek and B. Hajek, "Revenue optimal auction for single-minded buyers," in *49th IEEE Conference on Decision and Control (CDC)*. IEEE, 2010, pp. 1842–1847.
- [22] T. Borgers, R. Strausz, and D. Krahmer, *An introduction to the theory of mechanism design*. Oxford University Press, USA, 2015.
- [23] M. M. Pai and R. Vohra, "Optimal dynamic auctions and simple index rules," *Mathematics of Operations Research*, vol. 38, no. 4, pp. 682–697, 2013.
- [24] A. Nayyar, M. Negrete-Pincetic, K. Poola, and P. Varaiya, "Duration-differentiated energy services with a continuum of loads," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 2, pp. 182–191, June 2016.
- [25] P. R. Milgrom, *Putting auction theory to work*. Cambridge University Press, 2004.